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# Quantum spin chain with 'soliton non-preserving' boundary conditions

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**Abstract.** We consider the case of an integrable quantum spin chain with 'soliton non-preserving' boundary conditions. This is the first time that such boundary conditions have been considered in the spin chain framework. We construct the transfer matrix of the model, we study its symmetry and we find explicit expressions for its eigenvalues. Moreover, we derive a new set of Bethe ansatz equations by means of the analytical Bethe ansatz method.

## 1. Introduction

So far, quantum spin chains with 'soliton preserving' boundary conditions have been studied [1–3]. However, there exists another type of boundary condition, namely the 'soliton non-preserving' ones. These conditions are basically known in affine Toda field theories [4–6], although there is already a hint of such boundary conditions in the prototype paper of Sklyanin [7], which is further clarified by Delius in [4]. It is important to mention that in affine Toda field theories only the 'soliton non-preserving' boundary conditions have been studied [6, 8]. It is still an open question what the 'soliton preserving' boundary conditions are in these theories.

In this paper we construct the open spin chain with the 'new' boundary conditions, we show that the model is integrable, we study its symmetry and, evidently, we solve it by means of the analytical Bethe ansatz method [9–11]. This is the first time that such boundary conditions have been considered in the spin chain framework.

To describe the model it is necessary to introduce the basic constructing elements, namely, the $R$ and $K$ matrices.

The $R$ matrix is a solution of the Yang–Baxter equation

$$R_{12}(\lambda_1 - \lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1 - \lambda_2) \tag{1.1}$$

(see, e.g., [12]).

Here, we focus on the special case of the $SU(3)$ invariant $R$ matrix [13]:

$$\begin{aligned} &R_{12}(\lambda)_{jj,jj} = (\lambda + i) && \\ &R_{12}(\lambda)_{jk,jk} = \lambda && j \neq k \\ &R_{12}(\lambda)_{jk,kj} = i && j \neq k \\ & && 1 \leqslant j \quad k \leqslant 3. \end{aligned} \tag{1.2}$$

We also need to introduce the $R$ matrix that involves different representations of $SU(3)$ [14, 15], in particular, 3 and $\bar{3}$ (see also [16]). This matrix is given by crossing [17–20]

$$R_{\bar{1}2}(\lambda) = V_1\, R_{12}(-\lambda-\rho)^{t_2}\, V_1 = V_2^{t_2}\, R_{12}(-\lambda-\rho)^{t_1}\, V_2^{t_2} \tag{1.3}$$

where $V^2 = 1$. Note that

$$R_{\bar{i}j}(\lambda) = R_{i\bar{j}}(\lambda) \equiv \bar{R}_{ij}(\lambda) \qquad R_{\bar{i}\bar{j}}(\lambda) = R_{ij}(\lambda). \tag{1.4}$$

The $\bar{R}$ matrix is also a solution of the Yang–Baxter equation

$$\bar{R}_{12}(\lambda_1-\lambda_2)\bar{R}_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)\bar{R}_{13}(\lambda_1)\bar{R}_{12}(\lambda_1-\lambda_2). \tag{1.5}$$

The matrices $K^-$ and $K^+$ are solutions of the boundary Yang–Baxter equation [6, 21]

$$R_{12}(\lambda_1-\lambda_2)K_1^-(\lambda_1)\bar{R}_{21}(\lambda_1+\lambda_2)K_2^-(\lambda_2) = K_2^-(\lambda_2)\bar{R}_{12}(\lambda_1+\lambda_2)K_1^-(\lambda_1)R_{21}(\lambda_1-\lambda_2) \tag{1.6}$$

and

$$\begin{aligned} &R_{12}(-\lambda_1+\lambda_2)K_1^+(\lambda_1)^{t_1}\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)K_2^+(\lambda_2)^{t_2} \\ &\quad = K_2^+(\lambda_2)^{t_2}\bar{R}_{12}(-\lambda_1-\lambda_2-2\rho)K_1^+(\lambda_1)^{t_1}R_{21}(-\lambda_1+\lambda_2) \end{aligned} \tag{1.7}$$

where $\rho = \frac{3\mathrm{i}}{2}$. We can consider that the $K_i$ matrix describes the reflection of a soliton with the boundary which comes back as an anti-soliton (see also [6]).

It is a natural choice to consider the following alternating spin chain [14, 15], which leads to a local Hamiltonian. The corresponding transfer matrix $t(\lambda)$ for the open chain of $2N$ sites with 'soliton non-preserving' boundary conditions is (see also, e.g., [7, 22])

$$t(\lambda) = \mathrm{tr}_0\, K_0^+(\lambda)T_0(\lambda)K_0^-(\lambda)\hat{T}_{\bar{0}}(\lambda) \tag{1.8}$$

where $\mathrm{tr}_0$ denotes the trace over the 'auxiliary space' 0 and $T_0(\lambda)$ is the monodromy matrix,

$$\begin{aligned} T_0(\lambda) &= R_{02N}(\lambda)\bar{R}_{02N-1}(\lambda)\ldots R_{02}(\lambda)\bar{R}_{01}(\lambda) \\ \hat{T}_{\bar{0}}(\lambda) &= R_{10}(\lambda)\bar{R}_{20}(\lambda)\ldots R_{2N-10}(\lambda)\bar{R}_{2N0}(\lambda). \end{aligned} \tag{1.9}$$

We can change the auxiliary space to its conjugate and then we obtain the $\bar{t}(\lambda)$ matrix, which satisfies, for $K^{\pm}(\lambda) = 1$,

$$\bar{t}(\lambda) = t(\lambda)^t. \tag{1.10}$$

In particular,

$$\bar{t}(\lambda) = \mathrm{tr}_0\, K_{\bar{0}}^+(\lambda)T_{\bar{0}}(\lambda)K_{\bar{0}}^-(\lambda)\hat{T}_0(\lambda) \tag{1.11}$$

with

$$\begin{aligned} T_{\bar{0}}(\lambda) &= \bar{R}_{02N}(\lambda)R_{02N-1}(\lambda)\ldots \bar{R}_{02}(\lambda)R_{01}(\lambda) \\ \hat{T}_0(\lambda) &= \bar{R}_{10}(\lambda)R_{20}(\lambda)\ldots \bar{R}_{2N-10}(\lambda)R_{2N0}(\lambda) \end{aligned} \tag{1.12}$$

(we usually suppress the 'quantum-space' subscripts $1,\ldots,N$). One can observe the alternation between $R$ and $\bar{R}$ in (1.9) and (1.12). In particular for the monodromy matrix $T_0$ we see that in even sites there exists the $R$ matrix whereas in the odd sites the $\bar{R}$ matrix acts. The situation is exactly the opposite for the $\hat{T}_{\bar{0}}$ matrix. In fact,

$$\hat{T}_a(\lambda) = T_a^{-1}(-\lambda) \tag{1.13}$$

where $a$ can be 0 or $\bar{0}$. In the above definitions of the monodromy matrices we used the equations (1.4).

The transfer matrix satisfies the commutativity property

$$[t(\lambda), t(\lambda')] = 0. \tag{1.14}$$

$\bar{t}$ also obeys the commutativity property,

$$[\bar{t}(\lambda), \bar{t}(\lambda')] = 0; \tag{1.15}$$

moreover

$$[\bar{t}(\lambda), t(\lambda')] = 0. \tag{1.16}$$

We give a detailed proof of (1.14)–(1.16) in appendix A. The corresponding open spin chain Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} \propto \frac{\mathrm{d}}{\mathrm{d}\lambda} t(\lambda)\bar{t}(\lambda)\bigg|_{\lambda=0}. \tag{1.17}$$

It is necessary to consider the product of the two transfer matrices in order to obtain a local theory. One can show that this Hamiltonian is indeed local with terms that describe interaction up to four neighbours (see appendix B).

The outline of this paper is as follows: in the next section we briefly discuss the crossing symmetry of the transfer matrix and the fusion for the $K$ matrices and the transfer matrix. In section 3 we study the asymptotic behaviour and the symmetry of the transfer matrix. We show that, although we build the chain using the $SU(3)$ invariant $R$ matrix, the model has $SO(3)$ symmetry. In the following section we find the exact expressions for the transfer matrix eigenvalues and we also deduce a completely new set of Bethe ansatz equations via the analytical Bethe ansatz method. Finally, in the last section we review the results of this work and we also discuss some of our future goals.

## 2. Crossing and fusion

In this section we basically review known ideas about the crossing and the fusion procedure for the $R$ and $K$ matrices (see, e.g., [16,23,24]). We can prove (see also [10]) that the transfer matrix satisfies the crossing symmetry. To do this we need the identity

$$\mathcal{P}_{12}^{t_2}\bar{R}_{21}(\lambda)^{t_1} = \bar{R}_{21}(\lambda)^{t_1}\mathcal{P}_{12}^{t_2} \tag{2.1}$$

where $\mathcal{P}$ is the permutation operator. The last equation follows from the reflection equation (1.6) for $\lambda_1 - \lambda_2 = -\rho$, which follows from the reflection equation (1.6) for $\lambda_1 - \lambda_2 = -\rho$. Then we can show for the transfer matrix that

$$t(\lambda) = t(-\lambda - \rho). \tag{2.2}$$

Indeed, the transfer matrix does have crossing symmetry.

The fused $R$ matrices are known (see e.g., [16]). However, we still need to fuse the $K$ matrices. We consider the following reflection equation for $\lambda_1 - \lambda_2 = -\rho$:

$$\bar{R}_{12}(\lambda_1 - \lambda_2)K_{\bar{1}}^{-}(\lambda_1)R_{21}(\lambda_1 + \lambda_2)K_{2}^{-}(\lambda_2) = K_{2}^{-}(\lambda_2)R_{12}(\lambda_1 + \lambda_2)K_{\bar{1}}^{-}(\lambda_1)\bar{R}_{21}(\lambda_1 - \lambda_2) \tag{2.3}$$

then the fused $K$ matrices are given by

$$\begin{aligned} K_{\langle\bar{1}2\rangle}^{-}(\lambda) &= P_{\bar{1}2}^{+}K_{\bar{1}}^{-}(\lambda)R_{21}(2\lambda+\rho)K_{2}^{-}(\lambda+\rho)P_{2\bar{1}}^{+} \\ K_{\langle\bar{1}2\rangle}^{+}(\lambda)^{t_{12}} &= P_{2\bar{1}}^{+}K_{\bar{1}}^{+}(\lambda)^{t_1}R_{21}(-2\lambda-3\rho)K_{2}^{+}(\lambda+\rho)^{t_2}P_{\bar{1}2}^{+} \end{aligned} \tag{2.4}$$

where

$$P_{\bar{1}2}^{+} = 1 - \tfrac{1}{3}\bar{R}_{12}(-\rho) \tag{2.5}$$

is a projector to an eight-dimensional subspace (see also [16]) ($\frac{1}{3}\bar{R}_{12}(-\rho)$ is a projector to a one-dimensional subspace). Analogously, we obtain the $K_{\langle 1\bar{2}\rangle}(\lambda)$ matrices. The above $K$

matrices obey generalized reflection equations (see, e.g., [16,24]). One can show, for the case where $K^{\pm}(\lambda) = 1$, that the fused transfer matrix is (see, e.g., [24])

$$\hat{t}(\lambda) = \zeta'(2\lambda + 2\rho)\bar{t}(\lambda)t(\lambda + \rho) - \zeta(\lambda + \rho)^N \zeta'(\lambda + \rho)^N g(2\lambda + \rho)g(-2\lambda - 3\rho) \tag{2.6}$$

where we define

$$g(\lambda) = \lambda + \mathrm{i} \qquad \zeta(\lambda) = (\lambda + \mathrm{i})(-\lambda + \mathrm{i}) \qquad \zeta'(\lambda) = (\lambda + \rho)(-\lambda + \rho). \tag{2.7}$$

Note that we obtain one equation from fusion whereas in [16] we end up with two such equations.

## 3. The symmetry of the transfer matrix

Here, we study the symmetry of the transfer matrix for the alternating spin chain. To do so it is necessary to derive the asymptotic behaviour of the monodromy matrix. The asymptotic behaviour of the $R$, $\bar{R}$ matrices for $\lambda \to \infty$ follows from (1.2), (1.3):

$$\begin{aligned} R_{0k}(\lambda) &\sim \lambda \left( I + \frac{\mathrm{i}}{\lambda} \begin{pmatrix} S_{1,k} & J^-_{1,k} & J^-_{3,k} \\ J^+_{1,k} & S_{2,k} & J^-_{2,k} \\ J^+_{3,k} & J^+_{2,k} & S_{3,k} \end{pmatrix} \right) \\ \bar{R}_{0k}(\lambda) &\sim -\lambda \left( I + \frac{3\mathrm{i}}{2\lambda} I - \frac{\mathrm{i}}{\lambda} \begin{pmatrix} S_{3,k} & -J^-_{2,k} & J^-_{3,k} \\ -J^+_{2,k} & S_{2,k} & -J^-_{1,k} \\ J^+_{3,k} & -J^+_{1,k} & S_{1,k} \end{pmatrix} \right). \end{aligned} \tag{3.1}$$

The matrix elements are

$$\begin{aligned} &S_i = e_{i,i} \qquad i = 1, 2, 3 \\ &J^+_i = e_{i,i+1} \qquad J^-_i = e_{i+1,i} \qquad i = 1, 2 \\ &J^+_3 = e_{1,3} \qquad J^-_3 = e_{3,1} \end{aligned} \tag{3.2}$$

with

$$(e_{i,j})_{kl} = \delta_{ik}\delta_{jl}. \tag{3.3}$$

The leading asymptotic behaviour of the monodromy matrix is given by

$$\begin{aligned} T^+ &\sim (-)^{\frac{N}{2}} \lambda^N \left( I + \frac{3N\mathrm{i}}{2\lambda} I + \frac{\mathrm{i}}{\lambda} \begin{pmatrix} \mathcal{S}^{\mathrm{e}}_1 - \mathcal{S}^{\mathrm{o}}_3 & \mathcal{J}^{-\mathrm{e}}_1 + \mathcal{J}^{-\mathrm{o}}_2 & \mathcal{J}^{-\mathrm{e}}_3 - \mathcal{J}^{-\mathrm{o}}_3 \\ \mathcal{J}^{+\mathrm{e}}_1 + \mathcal{J}^{+\mathrm{o}}_2 & \mathcal{S}^{\mathrm{e}}_2 - \mathcal{S}^{\mathrm{o}}_2 & \mathcal{J}^{-\mathrm{e}}_2 + \mathcal{J}^{-\mathrm{o}}_1 \\ \mathcal{J}^{+\mathrm{e}}_3 - \mathcal{J}^{+\mathrm{o}}_3 & \mathcal{J}^{+\mathrm{e}}_2 + \mathcal{J}^{+\mathrm{o}}_1 & \mathcal{S}^{\mathrm{e}}_3 - \mathcal{S}^{\mathrm{o}}_1 \end{pmatrix} \right) \\ \hat{T}^+ &\sim (-)^{\frac{N}{2}} \lambda^N \left( I + \frac{3N\mathrm{i}}{2\lambda} I + \frac{\mathrm{i}}{\lambda} \begin{pmatrix} \mathcal{S}^{\mathrm{o}}_1 - \mathcal{S}^{\mathrm{e}}_3 & \mathcal{J}^{-\mathrm{o}}_1 + \mathcal{J}^{-\mathrm{e}}_2 & \mathcal{J}^{-\mathrm{o}}_3 - \mathcal{J}^{-\mathrm{e}}_3 \\ \mathcal{J}^{+\mathrm{o}}_1 + \mathcal{J}^{+\mathrm{e}}_2 & \mathcal{S}^{\mathrm{o}}_2 - \mathcal{S}^{\mathrm{e}}_2 & \mathcal{J}^{-\mathrm{o}}_2 + \mathcal{J}^{-\mathrm{e}}_1 \\ \mathcal{J}^{+\mathrm{o}}_3 - \mathcal{J}^{+\mathrm{e}}_3 & \mathcal{J}^{+\mathrm{o}}_2 + \mathcal{J}^{+\mathrm{e}}_1 & \mathcal{S}^{\mathrm{o}}_3 - \mathcal{S}^{\mathrm{e}}_1 \end{pmatrix} \right) \end{aligned} \tag{3.4}$$

where the superscripts e and o refer to the sum over even and odd sites of the chain respectively, namely,

$$\mathcal{S}^r_i = \sum_{k=[r]} S_{i,k} \qquad \mathcal{J}^{\pm r}_i = \sum_{k=[r]} J^{\pm}_{i,k} \qquad i = 1, 2, 3 \tag{3.5}$$

$r$ can be even or odd. To determine the symmetry of the transfer matrix we need the asymptotic behaviour of the following product:

$$T^+\hat{T}^+ \sim \lambda^{2N} \left( I + \frac{3N\mathrm{i}}{\lambda} I + \frac{\mathrm{i}}{\lambda} \begin{pmatrix} \mathcal{S} & \mathcal{J}^- & 0 \\ \mathcal{J}^+ & 0 & \mathcal{J}^- \\ 0 & \mathcal{J}^+ & -\mathcal{S} \end{pmatrix} \right) \tag{3.6}$$

where

$$\mathcal{S} = \mathcal{S}_1 - \mathcal{S}_3 \qquad \mathcal{J}^{\pm} = \mathcal{J}^{\pm}_1 + \mathcal{J}^{\pm}_2 \tag{3.7}$$

are the generators of $SO(3)$, and

$$\mathcal{S}_i = \mathcal{S}_i^{\mathrm{e}} + \mathcal{S}_i^{\mathrm{o}} \qquad \mathcal{J}_i^{\pm} = \mathcal{J}_i^{\pm\mathrm{e}} + \mathcal{J}_i^{\pm\mathrm{o}}. \tag{3.8}$$

We define the following operator, which has a structure similar to the transfer matrix:

$$\tau = \mathrm{tr}_0\, PT^{+}\hat{T}^{+} \tag{3.9}$$

where $P$ can be $S$, $J^{\pm}$ and projects out the corresponding generators from the (3.6). One can prove (see also [3]) the commutation relation

$$[t(\lambda), \tau] = 0. \tag{3.10}$$

Similarly, one can show that $\bar{t}(\lambda)$ commutes with $\tau$, therefore the Hamiltonian (1.17) commutes with $\tau$ as well. It is manifest from the equation (3.10) that the transfer matrix ($\bar{t}(\lambda)$ as well) has $SO(3)$ symmetry. Even though the result seems 'bizarre', it is somehow expected if we consider that $SO(3)$ is a subalgebra of $SU(3)$ invariant under charge conjugation. Remember that we constructed the spin chain which involves the 3 and $\bar{3}$ representations of $SU(3)$ in both quantum and auxiliary spaces. Moreover, it is essential for the following to determine the asymptotic behaviour of the transfer matrix eigenvalue, which is given by

$$t(\lambda) \sim \lambda^{2N}\left(3 + \frac{9N\mathrm{i}}{\lambda}\right) I \tag{3.11}$$

where $I$ is the $3 \times 3$ unit matrix.

## 4. Bethe ansatz equations

We can use the results of the previous sections in order to deduce the Bethe ansatz equations for the spin chain. First, we have to derive a reference state, namely the pseudo-vacuum. We consider the state with all spins up, i.e.

$$|\Lambda^{(0)}\rangle = \bigotimes_{k=1}^{N} |+\rangle_{(k)}; \tag{4.1}$$

this is annihilated by $\mathcal{J}^{+}$ where (we suppress the $(k)$ index)

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

This is an eigenstate of the transfer matrix. The action of the $R$, $\bar{R}$ matrices on the $|+\rangle$ ($\langle +|$) state gives upper (lower) triangular matrices. Consequently, the action of the monodromy matrix on the pseudo-vacuum also gives upper (lower) triangular matrices (see also [16]). We find that the transfer matrix eigenvalue for the pseudo-vacuum state, after some tedious calculations, is

$$\Lambda^{(0)}(\lambda) = (a(\lambda)\bar{b}(\lambda))^{2N}\frac{2\lambda + \frac{\mathrm{i}}{2}}{2\lambda + \frac{3\mathrm{i}}{2}} + (b(\lambda)\bar{b}(\lambda))^{2N} + (\bar{a}(\lambda)b(\lambda))^{2N}\frac{2\lambda + \frac{5\mathrm{i}}{2}}{2\lambda + \frac{3\mathrm{i}}{2}}. \tag{4.3}$$

Because of the $SO(3)$ symmetry of the transfer matrix there exist simultaneous eigenstates of $M = \frac{1}{2}(2N - S)$ and the transfer matrix, namely,

$$M|\Lambda^{(m)}\rangle = m|\Lambda^{(m)}\rangle \qquad t(\lambda)|\Lambda^{(m)}\rangle = \Lambda^{(m)}(\lambda)|\Lambda^{(m)}\rangle. \tag{4.4}$$

We assume that a general eigenvalue has the form of a 'dressed' pseudo-vacuum eigenvalue, i.e.

$$\Lambda^{(m)}(\lambda) = (a(\lambda)\bar{b}(\lambda))^{2N}\frac{2\lambda + \frac{\mathrm{i}}{2}}{2\lambda + \frac{3\mathrm{i}}{2}}A_1(\lambda) + (b(\lambda)\bar{b}(\lambda))^{2N}A_2(\lambda) + (\bar{a}(\lambda)b(\lambda))^{2N}\frac{2\lambda + \frac{5\mathrm{i}}{2}}{2\lambda + \frac{3\mathrm{i}}{2}}A_3(\lambda). \tag{4.5}$$

Our task is to find explicit expressions for the $A_i(\lambda)$. We consider all the conditions we derived previously. The asymptotic behaviour of the transfer matrix (3.11) gives the following condition for $\lambda \to \infty$:

$$\sum_{i=1}^{3} A_i(\lambda) \to 3. \tag{4.6}$$

The fusion equation (2.6) gives us conditions involving $A_1(\lambda)$, $A_3(\lambda)$, namely,

$$A_1(\lambda + \rho)A_3(\lambda) = 1. \tag{4.7}$$

The crossing symmetry of the transfer matrix (2.2) provides further restrictions among the dressing functions, i.e.

$$A_3(-\lambda - \rho) = A_1(\lambda) \qquad A_2(\lambda) = A_2(-\lambda - \rho). \tag{4.8}$$

The last two equations combined give

$$A_1(\lambda)A_1(-\lambda) = 1. \tag{4.9}$$

Moreover, for $\lambda = -\mathrm{i}$ the $R$ matrix degenerates to a projector onto a three-dimensional subspace. Thus, we can obtain another equation that involves $A_1(\lambda)$ and $A_2(\lambda)$ (see also [9]), namely,

$$A_2(\lambda)A_1(\lambda + \mathrm{i}) = A_1\left(\lambda + \frac{\mathrm{i}}{2}\right). \tag{4.10}$$

Finally, we require $A_2(\lambda)$ to have the same poles as $A_1(\lambda)$ and $A_3(\lambda)$. Considering all the above conditions together we find that

$$A_1(\lambda) = \prod_{j=1}^{m} \frac{\lambda + \lambda_j - \frac{\mathrm{i}}{2}}{\lambda + \lambda_j + \frac{\mathrm{i}}{2}} \frac{\lambda - \lambda_j - \frac{\mathrm{i}}{2}}{\lambda - \lambda_j + \frac{\mathrm{i}}{2}} \tag{4.11}$$

$$A_2(\lambda) = \prod_{j=1}^{m} \frac{\lambda + \lambda_j + \frac{3\mathrm{i}}{2}}{\lambda + \lambda_j + \frac{\mathrm{i}}{2}} \frac{\lambda - \lambda_j + \frac{3\mathrm{i}}{2}}{\lambda - \lambda_j + \frac{\mathrm{i}}{2}} \frac{\lambda + \lambda_j}{\lambda + \lambda_j + \mathrm{i}} \frac{\lambda - \lambda_j}{\lambda - \lambda_j + \mathrm{i}} \tag{4.12}$$

$$A_3(\lambda) = \prod_{j=1}^{m} \frac{\lambda + \lambda_j + 2\mathrm{i}}{\lambda + \lambda_j + \mathrm{i}} \frac{\lambda - \lambda_j + 2\mathrm{i}}{\lambda - \lambda_j + \mathrm{i}}. \tag{4.13}$$

We can check that the above functions indeed satisfy all the necessary properties. Finally, the analyticity of the eigenvalues (the poles must vanish) provides the Bethe ansatz equations

$$e_1(\lambda_i)^{2N} e_{-1}(2\lambda_i) = -\prod_{j=1}^{m} e_2(\lambda_i - \lambda_j) e_2(\lambda_i + \lambda_j) e_{-1}(\lambda_i - \lambda_j) e_{-1}(\lambda_i + \lambda_j) \tag{4.14}$$

where we have defined $e_n(\lambda)$ as

$$e_n(\lambda) = \frac{\lambda + \frac{\mathrm{i}n}{2}}{\lambda - \frac{\mathrm{i}n}{2}}. \tag{4.15}$$

Notice that we obtain a completely new set of Bethe equations starting with the known $SU(3)$ invariant $R$ matrix. At this point we can make the following interesting observation. Consider the Bethe ansatz equations for the alternating spin chain with periodic boundary conditions, with $2N$ sites [15]

$$\begin{aligned} e_1(\lambda_i^{(1)})^{N_0} &= \prod_{i \neq j=1}^{m_1} e_2(\lambda_i^{(1)} - \lambda_j^{(1)}) \prod_{j=1}^{m_2} e_{-1}(\lambda_i^{(1)} - \lambda_j^{(2)}) \\ e_1(\lambda_i^{(2)})^{N_0^*} &= \prod_{i \neq j=1}^{m_2} e_2(\lambda_i^{(2)} - \lambda_j^{(2)}) \prod_{j=1}^{m_1} e_{-1}(\lambda_i^{(2)} - \lambda_j^{(1)}) \end{aligned} \tag{4.16}$$

$(N_0 + N_0^* = 2N)$. For the special case where $N_0 = N_0^* = N$, $m_1 = m_2$ and $\lambda_j^{(1)} = \lambda_j^{(2)}$, the previous equations become

$$e_1(\lambda_i)^N = -\prod_{j=1}^{m} e_2(\lambda_i - \lambda_j)e_{-1}(\lambda_i - \lambda_j). \tag{4.17}$$

The last equations are exactly 'halved' compared to (4.14). For the moment we do not have any satisfactory explanation for the significance of this coincidence.

Our results can probably be generalized for the spin chain constructed by the $SU(\mathcal{N})$ invariant $R$ matrix. We expect a reduced symmetry for the general case as well.

## 5. Discussion

We have constructed a quantum spin chain with 'soliton non-preserving' boundary conditions. Although we started with the $SU(3)$ invariant $R$ matrix, we have shown that the model has $SO(3)$ invariance (3.10). We have used this symmetry to find the spectrum of the transfer matrix and we have also deduced the Bethe ansatz equations (4.14) via the analytical Bethe ansatz method. It would be of great interest to study the trigonometric case. Hopefully, one can find diagonal solutions for the $K$ matrices and solve the trigonometric open spin chain. The interesting aspect of the trigonometric case is that one can possibly relate the lattice model to some boundary field theory. Indeed, we know that, for example, the critical periodic $A_{\mathcal{N}-1}^{(1)}$ spin chain can be regarded as a discretization of the corresponding affine Toda field theory [25]. Finally, one can presumably generalize the above construction using any $SU(\mathcal{N})$ invariant $R$ matrix. We hope to report on these issues in a future work [26].

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## Appendix A

In this section we are going to prove the integrability of the model, namely the commutation relations (1.14), (1.15) for the transfer matrices. We define the following operator which originally introduced by Sklyanin [7]:

$$\mathcal{T}_0^-(\lambda) = T_0(\lambda)K_0^-(\lambda)\hat{T}_{\bar{0}}(\lambda). \tag{A.1}$$

As we have already mentioned in the introduction the $K^-(\lambda)$ matrix satisfies the reflection equation (1.6), therefore the $\mathcal{T}_0^-(\lambda)$ operator obeys the fundamental relation

$$R_{12}(\lambda_1 - \lambda_2)\mathcal{T}_1^-(\lambda_1)\bar{R}_{21}(\lambda_1 + \lambda_2)\mathcal{T}_2^-(\lambda_2) = \mathcal{T}_2^-(\lambda_2)\bar{R}_{12}(\lambda_1 + \lambda_2)\mathcal{T}_1^-(\lambda_1)R_{21}(\lambda_1 - \lambda_2). \tag{A.2}$$

Now we are ready to prove (1.14)

$$\begin{aligned} t(\lambda_1)t(\lambda_2) &= \mathrm{tr}_1\, K_1^+(\lambda_1)\mathcal{T}_1^-(\lambda_1)\,\mathrm{tr}_2\, K_2^+(\lambda_2)\mathcal{T}_2^-(\lambda_2) \\ &= \mathrm{tr}_1\, K_1^+(\lambda_1)^{t_1}\mathcal{T}_1^-(\lambda_1)^{t_1}\,\mathrm{tr}_2\, K_2^+(\lambda_2)\mathcal{T}_2^-(\lambda_2) \\ &= \mathrm{tr}_{12}\, K_1^+(\lambda_1)^{t_1}K_2^+(\lambda_2)\mathcal{T}_1^-(\lambda_1)^{t_1}\mathcal{T}_2^-(\lambda_2) \end{aligned} \tag{A.3}$$

we use the crossing unitarity of the $\bar{R}$ matrix, namely,

$$\bar{R}_{21}(-\lambda-2\rho)^{t_2}\bar{R}_{21}(\lambda)^{t_1}=\zeta(\lambda) \tag{A.4}$$

then the product $t(\lambda_1)t(\lambda_2)$ becomes

$$\begin{aligned}
&\zeta^{-1}(\lambda_1+\lambda_2)\,\mathrm{tr}_{12}\,K_1^+(\lambda_1)^{t_1}K_2^+(\lambda_2)\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)^{t_2}\bar{R}_{21}(\lambda_1+\lambda_2)^{t_1}\mathcal{T}_1^-(\lambda_1)^{t_1}\mathcal{T}_2^-(\lambda_2)\\
&\quad=\zeta^{-1}(\lambda_1+\lambda_2)\,\mathrm{tr}_{12}(K_1^+(\lambda_1)^{t_1}\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)K_2^+(\lambda_2)^{t_2})^{t_2}\\
&\qquad\times(\mathcal{T}_1^-(\lambda_1)\bar{R}_{21}(\lambda_1+\lambda_2)\mathcal{T}_2^-(\lambda_2))^{t_1}\\
&\quad=\zeta^{-1}(\lambda_1+\lambda_2)\,\mathrm{tr}_{12}(K_1^+(\lambda_1)^{t_1}\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)K_2^+(\lambda_2)^{t_2})^{t_{12}}\\
&\qquad\times\mathcal{T}_1^-(\lambda_1)\bar{R}_{21}(\lambda_1+\lambda_2)\mathcal{T}_2^-(\lambda_2)
\end{aligned} \tag{A.5}$$

using the unitarity of the $R$ matrix, i.e.

$$R_{21}(-\lambda)R_{12}(\lambda)=\zeta(\lambda) \tag{A.6}$$

we obtain the following expression for the product:

$$\begin{aligned}
&\zeta^{-1}(\lambda_1-\lambda_2)\zeta^{-1}(\lambda_1+\lambda_2)\,\mathrm{tr}_{12}(K_1^+(\lambda_1)^{t_1}\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)K_2^+(\lambda_2)^{t_2})^{t_{12}}\\
&\qquad\times R_{21}(-\lambda_1+\lambda_2)R_{12}(\lambda_1-\lambda_2)\mathcal{T}_1^-(\lambda_1)\bar{R}_{21}(\lambda_1+\lambda_2)\mathcal{T}_2^-(\lambda_2)\\
&\quad=\zeta^{-1}(\lambda_1-\lambda_2)\zeta^{-1}(\lambda_1+\lambda_2)\,\mathrm{tr}_{12}(R_{12}(-\lambda_1+\lambda_2)K_1^+(\lambda_1)^{t_1}\\
&\qquad\times\bar{R}_{21}(-\lambda_1-\lambda_2-2\rho)K_2^+(\lambda_2)^{t_2})^{t_{12}}R_{12}(\lambda_1-\lambda_2)\mathcal{T}_1^-(\lambda_1)\bar{R}_{21}(\lambda_1+\lambda_2)\mathcal{T}_2^-(\lambda_2).
\end{aligned} \tag{A.7}$$

Finally, with the help of equations (1.7) and (A.2) and repeating all the previous steps in reverse order we end up where the last expression is just $t(\lambda_2)t(\lambda_1)$. In order to show (1.15) we need to define the following operator by changing the auxiliary space to its conjugate in (A.1)

$$\mathcal{T}_{\bar{0}}^-(\lambda)=T_{\bar{0}}(\lambda)K_{\bar{0}}^-(\lambda)\hat{T}_0(\lambda). \tag{A.8}$$

$\mathcal{T}_{\bar{0}}^-(\lambda)$ satisfies the same fundamental relation (A.2) with $\mathcal{T}_0^-(\lambda)$ (remember (1.4)). Following exactly the same steps as before we can show that (1.15) is also true.

It is also necessary to prove (1.16). The steps of the proof are very similar to the previous case. The only difference is that this time we have to consider the reflection equation (2.3). Using (2.3) we can show the fundamental relation for $\mathcal{T}_{\bar{0}}^-(\lambda)$ and $\mathcal{T}_0^-(\lambda)$,

$$\bar{R}_{12}(\lambda_1-\lambda_2)\mathcal{T}_{\bar{1}}^-(\lambda_1)R_{21}(\lambda_1+\lambda_2)\mathcal{T}_2^-(\lambda_2)=\mathcal{T}_2^-(\lambda_2)R_{12}(\lambda_1+\lambda_2)\mathcal{T}_{\bar{1}}^-(\lambda_1)\bar{R}_{21}(\lambda_1-\lambda_2). \tag{A.9}$$

Following the same procedure as before and using the relations

$$R_{21}(-\lambda-2\rho)^{t_2}R_{21}(\lambda)^{t_1}=\zeta'(\lambda) \tag{A.10}$$

and

$$\bar{R}_{21}(-\lambda)\bar{R}_{12}(\lambda)=\zeta'(\lambda) \tag{A.11}$$

we can prove (1.16). This concludes our proof for the integrability of the model.

## Appendix B

In this appendix we show explicitly that the Hamiltonian of the open spin chain is local with terms that describe interaction up to four nearest neighbours. We focus here on the special case where $K^{\pm}(\lambda)=1$. We exploit the fact that $R_{ij}(0)=\mathcal{P}_{ij}$, then the transfer matrices become (we write for simplicity $\bar{R}_{ij}(0)=\bar{R}_{ij}$)

$$\begin{aligned}
t(0)&=\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}_{02N-1}\ldots\mathcal{P}_{02k}\bar{R}_{02k-1}\ldots\mathcal{P}_{02}\bar{R}_{01}\\
\bar{t}(0)&=\mathrm{tr}_0\,\bar{R}_{02N}\mathcal{P}_{02N-1}\ldots\bar{R}_{02k}\mathcal{P}_{02k-1}\ldots\bar{R}_{02}\mathcal{P}_{01}.
\end{aligned} \tag{B.1}$$

We move the permutation operators along the elements of the product and having in mind that

$$\mathrm{tr}_0\, \mathcal{P}_{02N}\bar{R}_{02N} \propto 1 \qquad \mathcal{P}_{ij}A_{ik}\mathcal{P}_{ij} = A_{jk} \tag{B.2}$$

where $A$ is any operator, we end up with

$$\begin{aligned} t(0) \propto\ & \bar{R}_{2N-1\,2N}\bar{R}_{2N-3\,2N-2}\ldots\bar{R}_{2k-1\,2k}\ldots\bar{R}_{12}\mathcal{P}_{24}\ldots\mathcal{P}_{2k-2\,2k}\ldots\mathcal{P}_{2N-2\,2N} \\ & \times\mathcal{P}_{2N-3\,2N-1}\ldots\mathcal{P}_{2k-3\,2k-1}\ldots\mathcal{P}_{13}\bar{R}_{23}\ldots\bar{R}_{2k-2\,2k-1}\ldots\bar{R}_{2N-2\,2N-1}\mathcal{P}_{12N} \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} \bar{t}(0) \propto\ & \mathcal{P}_{1\,2N}\bar{R}_{2N-2\,2N-1}\ldots\bar{R}_{2k-2\,2k-1}\ldots\bar{R}_{23}\mathcal{P}_{13}\ldots\mathcal{P}_{2k-3\,2k-1}\ldots\mathcal{P}_{2N-3\,2N-1} \\ & \times\mathcal{P}_{2N-2\,2N}\ldots\mathcal{P}_{2k-2\,2k}\ldots\mathcal{P}_{24}\bar{R}_{12}\ldots\bar{R}_{2j-1\,2j}\ldots\bar{R}_{2N-32N-2}\bar{R}_{2N-1\,2N}. \end{aligned} \tag{B.4}$$

We also need the derivative of the transfer matrix for $\lambda = 0$. It is sufficient to show the calculation for $\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\bar{t}(\lambda)$ (the product $t(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\bar{t}(\lambda)$ gives similar terms). Taking the derivative of the transfer matrix we obtain four different sums, because the derivative hits $R$, $\bar{R}$ of the monodromy matrix $T$ and $\hat{T}$ as well, namely

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\bigg|_{\lambda=0} =\ & \sum_{j=1}^{N}\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}_{02N-1}\ldots\bar{R}'_{02j-1}\ldots\mathcal{P}_{02}\bar{R}_{01}\mathcal{P}_{10}\bar{R}_{20}\ldots\mathcal{P}_{02N-1}\bar{R}_{2N0} \\ & +\sum_{j=1}^{N}\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}_{02N-1}\ldots R'_{02j}\ldots\bar{R}_{01}\mathcal{P}_{10}\bar{R}_{20}\ldots\mathcal{P}_{02N-1}\bar{R}_{2N0} \\ & +\sum_{j=1}^{N}\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}_{02N-1}\ldots\mathcal{P}_{02}\bar{R}_{01}\mathcal{P}_{10}\bar{R}_{20}\ldots\bar{R}'_{02j}\ldots\mathcal{P}_{02N-1}\bar{R}_{2N0} \\ & +\sum_{j=1}^{N}\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}_{02N-1}\ldots\mathcal{P}_{02}\bar{R}_{01}\mathcal{P}_{10}\bar{R}_{20}\ldots R'_{02j-1}\ldots\mathcal{P}_{02N-1}\bar{R}_{2N0} \end{aligned} \tag{B.5}$$

(the prime denotes the derivative with respect to $\lambda$). Again we move the permutation operators properly along the tensor product and we also consider (B.2) and $\mathrm{tr}_0\,\mathcal{P}_{02N}\bar{R}'_{02N} \propto 1$, and, finally, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\bigg|_{\lambda=0} \propto\ & \sum_{j=1}^{N}\bar{R}_{2N-1\,2N}\bar{R}_{2N-3\,2N-2}\ldots\bar{R}'_{2j-1\,2j}\ldots\bar{R}_{12}\mathcal{P}_{24}\ldots\mathcal{P}_{2N-2\,2N} \\ & \times\mathcal{P}_{2N-3\,2N-1}\ldots\mathcal{P}_{13}\bar{R}_{23}\ldots\bar{R}_{2k-2\,2k-1}\ldots\bar{R}_{2N-2\,2N-1}\mathcal{P}_{1\,2N} \\ & +\sum_{j=1}^{N-1}\bar{R}_{2N-1\,2N}\bar{R}_{2N-3\,2N-2}\ldots\bar{R}_{2j+1\,2j+2}\check{R}'_{2j\,2j+2}\bar{R}_{2j-1\,2j} \\ & \times\ldots\bar{R}_{12}\mathcal{P}_{24}\ldots\mathcal{P}_{2N-2\,2N} \\ & \times\mathcal{P}_{2N-3\,2N-1}\ldots\mathcal{P}_{13}\bar{R}_{23}\ldots\bar{R}_{2k-2\,2k-1}\ldots\bar{R}_{2N-2\,2N-1}\mathcal{P}_{1\,2N} \\ & +\sum_{j=1}^{N-1}\bar{R}_{2N-1\,2N}\bar{R}_{2N-3\,2N-2}\ldots\bar{R}_{2k-1\,2k}\ldots\bar{R}_{12}\mathcal{P}_{24}\ldots\mathcal{P}_{2N-2\,2N} \\ & \times\mathcal{P}_{2N-3\,2N-1}\ldots\mathcal{P}_{13}\bar{R}_{23}\ldots\bar{R}'_{2j\,2j+1}\ldots\bar{R}_{2N-2\,2N-1}\mathcal{P}_{1\,2N} \\ & +\sum_{j=1}^{N-2}\bar{R}_{2N-1\,2N}\bar{R}_{2N-3\,2N-2}\ldots\bar{R}_{2k-1\,2k}\ldots\bar{R}_{12}\mathcal{P}_{24}\ldots\mathcal{P}_{2N-2\,2N} \\ & \times\mathcal{P}_{2N-3\,2N-1}\ldots\mathcal{P}_{13}\bar{R}_{23}\ldots\bar{R}_{2j\,2j+1}\check{R}'_{2j+1\,2j+3}\bar{R}_{2j+2\,2j+3}\ldots\bar{R}_{2N-2\,2N-1}\mathcal{P}_{1\,2N} \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{tr}_0\, \check{R}'_{02N} \bar{R}_{2N-1\,2N} \bar{R}_{2N-3\,2N-2} \ldots \bar{R}_{2k-1\,2k} \ldots \bar{R}_{12} \mathcal{P}_{24} \ldots \mathcal{P}_{2N-2\,2N} \\
&\times \mathcal{P}_{2N-3\,2N-1} \ldots \mathcal{P}_{13} \bar{R}_{23} \ldots \bar{R}_{2k-2\,2k-1} \ldots \bar{R}_{2N-2\,2N-1} \mathcal{P}_{1\,2N} \mathcal{P}_{02N} \bar{R}_{02N} + t(0) \\
&+\bar{R}_{2N-1\,2N} \bar{R}_{2N-3\,2N-2} \ldots \bar{R}_{2k-1\,2k} \ldots \bar{R}_{12} \mathcal{P}_{24} \ldots \mathcal{P}_{2N-2\,2N} \\
&\times \mathcal{P}_{2N-3\,2N-1} \ldots \mathcal{P}_{13} \check{R}'_{3\,2N} \bar{R}_{23} \ldots \bar{R}_{2k-2\,2k-1} \ldots \bar{R}_{2N-2\,2N-1} \mathcal{P}_{1\,2N} \\
&+\bar{R}_{2N-1\,2N} \bar{R}_{2N-3\,2N-2} \ldots \bar{R}_{2k-1\,2k} \ldots \bar{R}_{12} \mathcal{P}_{24} \ldots \mathcal{P}_{2N-2\,2N} \\
&\times \mathcal{P}_{2N-3\,2N-1} \ldots \mathcal{P}_{13} \bar{R}_{23} \ldots \bar{R}_{2k-2\,2k-1} \ldots \bar{R}_{2N-2\,2N-1} \check{R}'_{1\,2N-1} \mathcal{P}_{1\,2N}
\end{aligned} \tag{B.6}
$$

where $\check{R}'_{ij} = \mathcal{P}_{ij} R'_{ij}$. The last four terms in (B.6) come from the second and the third sum for $j = N$ and from the last sum for $j = 0$ and $j = N - 1$, respectively. Combining (B.4), (B.6) and having in mind that $\mathcal{P}_{ij} A_{ik} \mathcal{P}_{ij} = A_{jk}$ and $\mathcal{P}^2_{ij}$, $\bar{R}^2_{ij} \propto 1$, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda} t(\lambda)\bar{t}(\lambda)\bigg|_{\lambda=0} &\propto \sum_{j=1}^{N} \bar{R}'_{2j-1\,2j} \bar{R}_{2j-1\,2j} + \sum_{j=1}^{N-1} \bar{R}_{2j+1\,2j+2} \check{R}'_{2j\,2j+2} \bar{R}_{2j+1\,2j+2} \\
&+ \sum_{j=1}^{N-1} \bar{R}_{2j+1\,2j+2} \bar{R}_{2j-1\,2j} \bar{R}'_{2j-1\,2j+2} \bar{R}_{2j-1\,2j+2} \bar{R}_{2j-1\,2j} \bar{R}_{2j+1\,2j+2} \\
&+ \sum_{j=1}^{N-1} \bar{R}_{2j+1\,2j+2} \bar{R}_{2j-1\,2j} \bar{R}_{2j-1\,2j+2} \check{R}'_{2j-1\,2j+1} \bar{R}_{2j-1\,2j+2} \bar{R}_{2j-1\,2j} \bar{R}_{2j+1\,2j+2} \\
&+ \mathrm{tr}_0\, \check{R}'_{02N} \bar{R}_{2N-1\,2N} \mathcal{P}_{02N-1} \bar{R}_{02N-1} \bar{R}_{2N-1\,2N} + t(0)\bar{t}(0) + \bar{R}_{12} \check{R}'_{12} \bar{R}_{12}.
\end{aligned} \tag{B.7}
$$

Notice that the first two terms of the last equation give exactly the Hamiltonian of the alternating spin chain constructed by De Vega and Woyanorovich (see e.g. [14, 15]). The last term of (B.6) is included in the fourth sum of (B.7) for $j = N - 1$. It is obvious from (B.3), (B.4) that $t(0)\bar{t}(0) \propto 1$. Equation (B.7) contains all the Hamiltonian's terms (remember $t(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\bar{t}(\lambda)$ has a similar form to (B.7)). We observe that the terms in (B.7) describe local interaction between two, three and four nearest neighbours. We conclude that this is indeed a local Hamiltonian.

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